On the Internal Sum of Positive Monoids (Mentor: Dr. Felix Gotti)

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Outline

General Notations

Some General Notation

- $\mathbb{Q}_{\geq r}$ denotes the set of rationals greater than or equal to r.
- $\mathbb{N}$ denotes the set of positive integers.
- $\mathbb{N}_0$ denotes the set of non-negative integers.
- $\mathbb{P}$ denotes the set of positive integer primes.

Definition of a Monoid

Definition. A monoid is a set M equipped with a binary operation $+$ satisfying

- $(a+b)+c = a+(b+c)$ for all $a, b, c \in M$ (associativity);
- there exists $e \in M$ such that $e+a = a+e = a$ for all $a \in M$.

Additionally,

- A monoid M is commutative if $a+b = b+a$ for all $a, b \in M$.
- A commutative monoid M is cancellative if $a+c = b+c$ implies $a = b$ for all $a, b, c \in M$.

For the rest of this presentation, we will assume that all monoids are commutative and cancellative. We will also write all monoids additively and use 0 to denote the identity element of any monoid M unless otherwise specified.

Definition. A subset N of M is called a submonoid of M if N contains the identity of M and is closed under $+$.

Invertiblility and Divisability

Definition. Let M be a monoid. An element $u \in M$ is invertible if there exists $u' \in M$ such that $u + u' = 0$.

Remark

- In any monoid the set of invertible elements contains the identity and forms a group.
- Only one element of any positive monoid is invertible - the identity.

Definition. For $a, b \in M$ for some monoid M, we say a divides b and b is divisible by a if there exists $c \in M$ such that $a + c = b$. We also write this divisibility as $a \mid_M b$.

Types of Monoids

Definition. We define the following types of monoids.

- An submonoid of $\mathbb{N}_0$ under addition is called a numerical monoid.
- An submonoid of $\mathbb{Q}_{\geq 0}$ under addition is called a Puiseux monoid.
- An submonoid of $\mathbb{R}_{\geq 0}$ under addition is called a positive monoid.
- A Puiseux monoid M is called a valuation monoid if for every $a, b \in M$ with $a < b$ we have $a \mid_M b$.

Examples

- $\mathbb{Q}_{\geq 0}$ is a monoid under standard addition, and $\mathbb{Q}_{\geq r} \cup \{0\}$ for any $r \in \mathbb{Q}_{\geq 0}$ are its submonoids. These monoids are all Puiseux monoids.
- The monoid $\{a + b\pi + c\sqrt{5} \mid a, b, c \in \mathbb{N}_0\}$ under standard addition is a positive monoid.

Generating Sets and Finitely Generated Monoids

Definition. Given an abelian group G and $S \subseteq G$, the submonoid generated by S, written as $\langle S \rangle$, is given by the following equivalent definitions:

- The smallest (under set inclusion) submonoid of G containing S.
- The set consisting of all finite sums of elements of S. That is, the set
$$\left\{ \sum_{i=1}^{n} s_i \;\middle|\; n \in \mathbb{N}_0, s_i \in S \text{ for all } 1 \leq i \leq n \right\}.$$
Note that in the case of $n = 0$, the empty sum gives 0.

We say S is a generating set of $\langle S \rangle$, and we say that a monoid is finitely generated if it has a finite generating set.

Examples

- We can check the Puiseux monoid $\left\langle \frac{1}{p} \;\middle|\; p \in \mathbb{P} \right\rangle$ is not finitely generated.
- The monoid of Gaussian integers in the first quadrant $\{a + bi \mid a, b \in \mathbb{N}_0\}$ under addition is finitely generated with generating set $\{1, i\}$.
- The monoid of positive integers under multiplication is generated by $\mathbb{P}$. It is not finitely generated.
- The monoid $\left\langle \frac{1}{2^n} \mid n \in \mathbb{N} \right\rangle$ is a valuation monoid.

Internal Sums

Definition. Let G be an abelian group, and let M_1 and M_2 be submonoids of G. The internal sum of M_1 and M_2 is the smallest submonoid of G containing both M_1 and M_2, which can be written as

$$M_1 + M_2 := \{a + b \mid a \in M_1, b \in M_2\}.$$

Proposition (easy to show)

In an abelian group G, let M_1 be a submonoid generated by S_1 and M_2 be a submonoid generated by S_2, then $M_1 + M_2$ is generated by $S_1 \cup S_2$.

Examples

- The internal sum (in $\mathbb{N}_0$) of numerical monoids $\langle 3 \rangle$ and $\langle 5 \rangle$ is $\langle 3, 5 \rangle$, which consists of 0, 3, 5, 6, and every integer greater than or equal to 8.
- The internal sum in $\mathbb{Z}$ of $\langle 1 \rangle$ and $\langle -1 \rangle$ is $\mathbb{Z}$.

Atomicity

Definition. Let M be a monoid.

- A non-invertible element $a \in M$ is an atom if there does not exist non-invertibles $b, c \in M$ such that $b + c = a$. The set of atoms of M is denoted by $\mathcal{A}(M)$.
- We call an element $m \in M$ atomic if it is invertible or is in the submonoid generated by $\mathcal{A}(M)$.
- The monoid M is atomic if every element of M is atomic.

Examples

- A finitely generated monoid is atomic.
- The monoid $\left\langle \frac{1}{p} \;\middle|\; p \in \mathbb{P} \right\rangle$ is atomic because one can show that each $\frac{1}{p}$ is an atom, which means the entire monoid is generated by atoms.
- The set $\mathbb{N}$ under multiplication is atomic with $\mathcal{A}(\mathbb{N}) = \mathbb{P}$.
- The Puiseux monoid $\left\langle \frac{1}{3}, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \dots \right\rangle$ is not atomic. Its only atom is $\frac{1}{3}$.

Strong Atomicity

Definition. A monoid M is called strongly atomic if every pair $a, b \in M$ (possibly satisfying $a = b$) has an atomic common divisor $d \in \mathcal{A}(M)$ such that $a - d$ and $b - d$ has no non-invertible common divisors.

Proposition (Anderson-Anderson-Zafrullah, 1990)

Every strongly atomic monoid is atomic.

Proof

- Let M be a strongly atomic monoid and let $a \in M$.
- Because M is strongly atomic, there exists atomic $d \in M$ such that $d \mid a$ and $a - d$ has no non-invertible divisors.
- Therefore $a - d$ must be invertible and a is associates with d, which means a is also atomic.
- Thus, M is atomic.

Important Example. The Grams' monoid is the Puiseux monoid

$$\left\langle \frac{1}{2^n p_n} \;\middle|\; n \in \mathbb{N}_0 \right\rangle,$$

where $(p_n)_{n \geq 0}$ is the increasing sequence with underlying set $\mathbb{P} \setminus \{2\}$.

Proposition (Gotti-Li, 2022)

The Grams' monoid is strongly atomic.

Proof. Let the Grams' monoid be denoted by G. Note that G is atomic.

- The valuation monoid $N = \left\langle \frac{1}{2^n} \;\middle|\; n \in \mathbb{N} \right\rangle$ is a submonoid of G.
- Each $b \in G$ has a unique finite canonical decomposition: $b = a + \sum_{n \in \mathbb{N}} \frac{c_n}{2^n p_n}$ where $a \in N$ and $0 \leq c_n \leq p_n$
- Let $b_1, b_2 \in G$ with canonical decomposition $b_1 = a_1 + \sum_{n \in \mathbb{N}} \frac{c_{1,n}}{2^n p_n}$ and $b_2 = a_2 + \sum_{n \in \mathbb{N}} \frac{c_{2,n}}{2^n p_n}$.
- Let $d = \min\{a_1, a_2\} + \sum_{n \in \mathbb{N}} \frac{\min\{c_{1,n}, c_{2,n}\}}{2^n p_n}$. Then $b_1 - d$ and $b_2 - d$ have no non-invertible common divisors.

Preservation of Atomicity under Internal Sum

Theorem (D-L-Z, 2024)

Let M and N be positive monoids such that N is finitely generated. Then the following statements hold.

- *If M is atomic, then $M + N$ is atomic.*
- *If M is strongly atomic, then $M + N$ is strongly atomic.*

Example. Let us use $\mathbb{Z} \times \mathbb{Z}$ instead of $\mathbb{Q}_0$ as our universe. Consider the following two submonoids of $\mathbb{Z} \times \mathbb{Z}$:

- The monoid $N := \mathbb{N}_0 \times \mathbb{N}_0$ is finitely generated with $\mathcal{A}(\mathbb{N}_0 \times \mathbb{N}_0) = \{(1,0),(0,1)\}$.
- The monoid The monoid $M := (\mathbb{Z} \times \mathbb{N}) \cup (0,0)$ is atomic with $\mathcal{A}(M) = \{(n,1) \mid n \in \mathbb{Z}\}$.

Their internal sum turns out to be simply $N \cup M$, whose only invertible element is $(0,0)$ and only atom is $(1,0)$. Thus, $N + M$ is not atomic.

Internal Sums and Monoid Algebras

Definition. Let M be a positive monoid.

- The monoid algebra of M over $\mathbb{Z}$, denoted by $\mathbb{Z}[M]$, consists of all polynomial expressions in an indeterminate x with exponents in M and integer coefficients (under polynomial-like addition and multiplication).
- The monoid algebra $\mathbb{Z}[M]$ is called atomic (resp., strongly atomic) provided that the multiplicative monoid consisting of all nonzero polynomial expressions of $\mathbb{Z}[M]$ is atomic (resp., strongly atomic).

Example. For $f := x^4 + x^{\frac{3}{2}}$ and $g := x^{\frac{3}{2}} + 1$ in the monoid algebra $\mathbb{Z}[\mathbb{Q}_{\geq 0}]$,

$$f + g = x^4 + 2x^{\frac{3}{2}} + 1 \quad \text{and} \quad f \cdot g = x^{4+\frac{3}{2}} + x^4 + x^3 + x^{\frac{3}{2}}.$$

Open Questions. Let M_1 and M_2 be positive monoids.

1. Is the monoid algebra $\mathbb{Z}[M_1 + M_2]$ necessarily atomic when the internal sum $M_1 + M_2$ and the monoid algebras $\mathbb{Z}[M_1]$ and $\mathbb{Z}[M_2]$ are all atomic?
2. Is the monoid algebra $\mathbb{Z}[M_1 + M_2]$ necessarily strongly atomic when the internal sum $M_1 + M_2$ and the monoid algebras $\mathbb{Z}[M_1]$ and $\mathbb{Z}[M_2]$ are all strongly atomic?

Preservation of Atomicity under Internal Sum

Definition. A monoid M is called antimatter if $\mathcal{A}(M) = \emptyset$; that is, if M has no atoms.

Example. We exhibit an atomic Puiseux monoid whose internal sum with the Grams' monoid, also an atomic monoid, is antimatter.

1. Let p_n denote the n-th odd prime, and set $a_n := \frac{1}{2^n p_n}$ for every $n \in \mathbb{N}$.
2. Take $f\colon \mathbb{N} \to \mathbb{N}$ to be an injective function such that $p_{f(n)} > 2^n p_n$ for all $n \in \mathbb{N}$.
3. For each $n \in \mathbb{N}$, we construct an infinite sequence $(b_i)_{i \geq 1}$ such that

$$b_1 := a_n - a_{f(n)} > \frac{1}{2} a_n \quad \text{and} \quad b_{i+1} := b_i - \frac{1}{2^{c_i}},$$

where $c_i \in \mathbb{N}$ is chosen such that $b_{i+1} > \frac{1}{2} a_n$. We call these terms $(b_i)_{i \geq 1}$ n-atoms.

Preservation of Atomicity under Internal Sum

Example (continued).

4. Let S be the union of the sets of n-atoms over all n, and let M be the Puiseux monoid generated by S.
5. By the constructions of the sequences $(b_i)_{i \geq 1}$, M is an atomic monoid whose set of atoms is exactly the union of the sets of n-atoms over $n \in \mathbb{N}$.
6. Let $M' := M + \langle a_n \mid n \in \mathbb{N} \rangle$. We argue that M' is antimatter. Note that M' is generated by the union of the sets of a_n and n-atoms for $n \in \mathbb{N}$. Now:
    1. Any atom a_n of the Grams' monoid, is divisible in M' by any n-atom and therefore cannot be an atom of M'.
    2. Any n-atom b_k, which is an atom of M, is divisible in M' by b_{k+1}, which means it cannot be an atom of M'.
7. Thus, none of the generators of M' are atoms, so M' is an antimatter monoid, despite being the internal sum of two atomic Puiseux monoids.

The Ascending Chain Condition on Principal Ideals (ACCP)

Definition. Let M be a monoid.

- A subset I of M is called an ideal if $I + M := \{b + m \mid b \in I \text{ and } m \in M\} \subseteq I$.
- An ideal of the form $b + M$ for some $b \in M$ is called principal.
- An element b of a monoid M satisfies the ACCP if every ascending chain of principal ideals starting at $b + M$ stabilizes.
- A monoid satisfies the ACCP if all of its elements satisfy the ACCP.

Example.

1. The monoid $\left\langle \frac{1}{p} \,\middle|\, p \in \mathbb{P} \right\rangle$ satisfies the ACCP.

Remark. All ACCP monoids are atomic.

Remark. In fact, all ACCP monoids are strongly atomic.

$$\text{ACCP} \Rightarrow \text{Strong Atomicity} \Rightarrow \text{Atomicity}$$

Preservation of ACCP under Internal Sum

Theorem (Geroldinger-Gotti, 2024)

Let M and N be Puiseux monoids such that N is finitely generated. If M satisfies the ACCP, then $M + N$ satisfies the ACCP.

Example: There exist two ACCP Puiseux monoids whose internal sum does not satisfy the ACCP.

1. Let $M_0 = \{0\} \cup \mathbb{Q}_{\geq 1}$ and $N_0 = \left\langle \frac{1}{p} \;\middle|\; p \in \mathbb{P} \right\rangle$, both of which are atomic monoids.
2. Observe that $\mathcal{A}(M_0) = [1, 2) \cap \mathbb{Q}$ and $\mathcal{A}(N_0) = \left\{ \frac{1}{p} \;\middle|\; p \in \mathbb{P} \right\}$.
3. Now, the monoid $M := M_0 + N_0$ satisfies $\mathcal{A}(M) = \left\{ \frac{1}{p} \;\middle|\; p \in \mathbb{P} \right\}$.
4. Thus, M is not atomic, and so cannot satisfy the ACCP.

The Almost ACCP

Definition. A monoid M satisfies the almost ACCP if

1. M is atomic, and
2. every nonempty finite subset S of $M/\{0\}$ has a common divisor d such that $s - d$ satisfies the ACCP for some $s \in S$.

Remark. Every ACCP monoid satisfies the almost ACCP.

Remark. Every almost ACCP monoid is strongly atomic.

$$\text{ACCP} \Rightarrow \text{Almost ACCP} \Rightarrow \text{Strong Atomicity} \Rightarrow \text{Atomicity}$$

Examples.

1. The Grams' monoid $G := \left\langle \frac{1}{2^n p_n} \;\middle|\; n \in \mathbb{N}_0 \right\rangle$ satisfies the almost ACCP but does not satisfy the ACCP, due to the strictly ascending chain of principal ideals $\left(\frac{1}{2^n} + G\right)_{n \geq 1}$.
2. The monoid $\left\langle \frac{1}{p_n p_{n+2}} \;\middle|\; n \in \mathbb{N} \right\rangle$, where p_n is the nth prime, is strongly atomic but does not satisfy the almost ACCP.

Greatest-Divisor Submonoids

Definition. Let M be a Puiseux monoid, and let N be a submonoid of M. For each $b \in M$, the greatest divisor of b in N is the unique element $d \in N$ satisfying:

1. $d \mid_M b$ and
2. if $d' \mid_M b$ for some $d' \in N$, then $d' \mid_M d$.

We say that N is a greatest-divisor submonoid of M provided that every element of M has a greatest divisor in N.

For each $b \in M$, we let $\text{gd}_N(b)$ denote the unique greatest divisor of b in N.

Back to the Grams' Monoid

Example. Let $G := \left\langle \frac{1}{2^n p_n} \;\middle|\; n \in \mathbb{N}_0 \right\rangle$ be the Grams' monoid. Consider the valuation submonoid $N := \left\langle \frac{1}{2^n} \;\middle|\; n \in \mathbb{N}_0 \right\rangle \subset G$.

1. Every element $b \in G$ has a unique, finite canonical decomposition as
$$b = a + \sum_{n \in \mathbb{N}_0} \frac{c_n}{2^n p_n}$$
for $a \in N$ and $0 \leq c_n < p_n$.
2. In fact, for any other finite decomposition
$$b = a' + \sum_{n \in \mathbb{N}_0} \frac{c_n'}{2^n p_n}$$
for $a' \in N$ and $c_n' \in \mathbb{N}_0$, we must have $a' \mid_G a$.
3. Hence, we see that $\mathrm{gd}_N(b) = a$.
4. Thus, N is a valuation greatest-divisor submonoid of the Grams' monoid.

The Moderate ACCP

Definition. An atomic Puiseux monoid M satisfies the moderate ACCP if there exists a submonoid $N \subset M$ such that:

1. N is a valuation greatest-divisor submonoid of M, and
2. $m - \mathrm{gd}_N(m)$ is ACCP for every $m \in M$.

Remark. Every moderate ACCP monoid satisfies the almost ACCP.

Suppose that Puiseux monoid M satisfies the moderate ACCP.

1. For every nonempty finite subset S of $M/\{0\}$, take $d \in N$ such that $d = \min\{\mathrm{gd}_N(s) \mid s \in S\}$.
2. Thus, there exists $s \in S$ such that $\mathrm{gd}_N(s) = d$.
3. For that choice of s, we see that $s - d$ satisfies the ACCP.
4. Hence, M satisfies the almost ACCP.

ACCP $\Rightarrow$ Moderate ACCP $\Longleftrightarrow$ Almost ACCP $\Rightarrow$ Strong Atomicity $\Rightarrow$ Atomicity

Thus, we see that the Grams' monoid satisfies the almost ACCP.

Preservation of almost/moderate ACCP under Internal Sum

Theorem (D-L-Z, 2024)

Let M and N be Puiseux monoids such that N is finitely generated. If M satisfies the moderate ACCP, then $M + N$ satisfies the moderate ACCP.

Open Question. Let M and N be Puiseux monoids such that N is finitely generated. If M satisfies the almost ACCP, must $M + N$ satisfy the almost ACCP?

Remark. We again cannot replace the condition of being finitely generated with the weaker condition of satisfying the almost or moderate ACCP, due to the same counterexample as before.

Factorization in Monoids

Definition. Let M be a monoid, and let m be an element of M. An (additive) factorization of m is defined as a formal sum of finitely many atoms of M equaling m.

Examples.

1. In the simplest numerical monoid $\mathbb{N}_0$, the only possible factorization of a positive integer k is $k = \underbrace{1 + \cdots + 1}_{k \text{ 1's}}$.
2. In the Puiseux monoid $\left\langle \frac{1}{p} \mid p \in \mathbb{P} \right\rangle$, factorizations of 1 include $1 = \frac{1}{2} + \frac{1}{2}, \frac{1}{3} + \frac{1}{3} + \frac{1}{3}, \ldots$

Bounded Factorization Monoids

Definition. Let M be a monoid.

1. The length of a factorization of $r \in M$ is it's number of atoms (counting repetitions).
2. The monoid M is called a bounded factorization monoid (BFM) if for all noninvertible elements $r \in M$ there is an upper bound on the length of all factorizations of r.

Example. The monoid $M := \left\langle \frac{n-1}{n} \mid n \in \mathbb{N} \right\rangle$ is a BFM, since the minimal nonzero element is $\frac{1}{2}$, so every element $r \in M$ will have at most $\lfloor 2r \rfloor$ atoms in its factorization.

Remark. Every positive monoid M with a minimal nonzero element (in other words, zero is not a limit point of M) is a BFM.

Length-Finite Factorization Monoids

Definition. A monoid M is called a length-finite factorization monoid (LFFM) if for each $r \in M$ and $\ell \in \mathbb{N}$, the number of factorizations of r with length ℓ is finite.

Remark. The properties of being LFFM and being BFM are not comparable. (i.e. There exist examples of BFMs that are not LFFMs, and vice versa.)

Examples.

1. Grams' monoid is an LFFM that is not a BFM.
2. The Puiseux monoid $M := \{0\} \cup \mathbb{Q}_{\geq 1}$ is a BFM that is not an LFFM. This is because $3 \in M = \left(\frac{3}{2} - \frac{1}{n}\right) + \left(\frac{3}{2} + \frac{1}{n}\right)$ has infinitely many factorizations of length 2.

Factorization Properties Satisfied by the Grams' Monoid

Example. Let $G := \left\langle \frac{1}{2^n p_n} \mid n \in \mathbb{N}_0 \right\rangle$ be the Grams' monoid. Then G is an LFFM that is not a BFM.

1. Every element r in Grams' monoid can be written uniquely as
$$r = \frac{n}{2^k} + \sum_{i=1}^{k} \frac{q_i}{2^i p_i} \mid n, k, q_i \in \mathbb{N}_0, q_i < 2^i p_i.$$
2. Now assume for the sake of contradiction that r has infinitely many factorizations of length ℓ.
3. Then the atoms $a = \frac{1}{2^a p_a}$ in these factorizations must satisfy $p_a \leq \max(p_k, \ell)$, where we recall p_k from the unique representation of r. There are only a finite number x of such atoms.
4. Thus the factorizations of r with length ℓ are all a sum with ℓ addends with x choices for each addend.
5. This results in at most x^ℓ factorizations, a contradiction.
6. G is not a BFM is more straightforward as there is no bound for the length of factorizations of 1.
7. Hence G is an LFFM that is not a BFM.

Finite Factorization Monoids

Definition. A monoid M is a finite factorization monoid (FFM) if every element $r \in M$ is only divisible by finitely many elements of M (up to associates).

Remark. A monoid is an FFM if and only if it is both an LFFM and a BFM.

Examples

1. Every finitely generated monoid, for example $\mathbb{N}_0$, which is generated by 1, is an FFM.
2. $M := \left\langle \frac{p-1}{p} \mid p \in \mathbb{P}_{\geq 5} \right\rangle$ is an example of an FFM that is clearly not finitely generated.

Preservation of Factorization Properties under Internal Sum

Theorem (D-L-Z, 2024)

Let M and N be positive monoids such that N is finitely generated. Then the following statements hold:

1. *If M is an FFM, then $M + N$ is an FFM.*
2. *If M is a BFM, then $M + N$ is a BFM.*
3. *If M is an LFFM, then $M + N$ is an LFFM.*

Remark. This is not the case when considering the internal sum of two FFMs, BFMs, or LFFMs.

Preservation of Factorization Properties under Internal Sum

Example. Let $M_a := \left\langle \frac{p-1}{p} \mid p \in \mathbb{P}_{\geq 5} \right\rangle$, and let $M_b := \left\langle \frac{p+1}{p} \mid p \in \mathbb{P}_{\geq 5} \right\rangle$. We argue that M_a and M_b are both FFMs, but their internal sum is not even an LFFM.

1. It is easy to see that the set of generators of M_a is $\mathcal{A}(M_a)$.
2. Similar to how we showed Grams' monoid is an LFFM, we can show that every element $r \in M_a$ can be written (uniquely) as $r = N(r) + \sum_{i=1}^{k} \frac{q_i(p_i - 1)}{p_i} \mid N(r), k, q_i \in \mathbb{N}_0, q_i < p_i$.
3. Only finitely many atoms divide the sum $\sum_{i=1}^{k} \frac{q_i(p_i-1)}{p_i}$.
4. An atom $\frac{p-1}{p}$ can only divide $N(r)$ if $N(r) \geq p - 1$, and thus only finitely many atoms can divide $N(r)$.
5. This implies that finitely many atoms divide r, which then implies that only finitely many elements divide r. Similarly, M_b is an FFM.

However, the internal sum $M = M_a + M_b$ contains 2, which has infinitely many factorizations factorizations of length 2, namely $\frac{p-1}{p} + \frac{p+1}{p}$ (which remain atoms in M), and thus M is not an LFFM.

Example. Let $M_1 := \{0\} \cup \mathbb{Q}_{\geq 1}$, and let $M_2 := \left\langle \frac{p+1}{p^2} \mid p \in \mathbb{P} \right\rangle$. We will show that M_1 and M_2 are both BFMs, but their internal sum is not even atomic.

1. As we have seen before, M_1 has a minimal nonzero element, so it is a BFM.
2. Mimicking the argument on the previous slide, we can show that M_2 is an FFM and thus a BFM.
3. It is easy to show that $\mathcal{A}(M_1) = [1, 2) \cap \mathbb{Q}$ and $\mathcal{A}(M_2) = \{ \frac{p+1}{p^2} \mid p \in \mathbb{P}\}$.
4. However, in their internal sum $M = M_1 + M_2$, for every atom $a_1 \in M_1$ except for 1, we can find a sufficiently small atom $a_2 \in M_2$ such that $a_2 \mid_M a_1$.
5. The atoms of M_2 remain atoms in M, so $\mathcal{A}(M_1 + M_2) = \mathcal{A}(M_2) \cup \{1\}$.
6. However, these atoms cannot sum to any fraction with a denominator divisible by the cube of a prime, such as $\frac{9}{8}$, which is clearly an element in M. Thus M is not atomic.

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End of Presentation

THANK YOU!